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Cosmic Lorentz Transformation

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In special relativity the Lorentz transformation gives the substitutions for distances and time measurements in two coordinate systems moving relative to each other with a constant velocity. We derive the corresponding transformation for cosmology that connects coordinate systems in different locations and measuring distances and red shifts.

In special relativity an event is described by three spatial coordinates x, y, z, and time t. Another observer with spatial coordinates x', y', z', and time t' can make the measurement of the same event. The special theory of relativity assumes that the two coordinate systems are inertial and move relative to each other with a constant speed. One then has the line element $x^2+y^2+z^2-c^2t^2$ invariant, namely (Einstein, 1955)

$$x^2+y^2+z^2-c^2t^2=x'^2+y'^2+z'^2-c^2t'^2 \tag{1}$$

The transformation that relates the unprimed and the primed variables in equation (1) is then the Lorentz transformation. For instance if the two coordinate systems (denoted by K and K') move relative to each other along the x axis (and coincide at $t=0$), then one has

$$x'=\frac{x-vt}{(1-v^2/c^2)^{1/2}}, \qquad t'=\frac{t-vx/c^2}{(1-v^2/c^2)^{1/2}} \tag{2}$$

along with $y'=y$ and $z'=z$. Equation (1), when equated to zero, expresses the fact that light propagates with the constant speed c in all inertial coordinate systems.

In cosmology the analog to the principle of relativity is the cosmological principle and the analog to the constancy of the speed of propagation of light is Hubble's law of the constancy of the expansion of the Universe.

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An event in cosmology is expressed by measurements of three spatial coordinates x, y, z, and the red shift parameter Z of the object. Hubble's law then tells us that the farther the object is from us the bigger its red shift, namely $r \propto Z$ or $r = aZ$, where a is a constant distance which can be taken as $a = cT$, with $T = 18 \times 10^9$ years. Thus in general one has the expression $x^2 + y^2 + z^2 - a^2Z^2$ invariant. The cosmological principle then tells us that the same measurement can equally be made by another observer whose coordinates are x', y', z', and Z', and that one has (Weinberg, 1972; Peebles, 1980)

$$x^2 + y^2 + z^2 - a^2Z^2 = x'^2 + y'^2 + z'^2 - a^2Z'^2 \tag{3}$$

where the rate of change of the spatial coordinates of one coordinate system relative to the other, with respect to the red shift parameter, is assumed to be a constant. Thus one sees that the role of time is being replaced in cosmology by the red shift parameter. For example if one assumes that the two coordinate systems are displaced relative to each other along the x axis and coincide at $Z = 0$, then one has the transformation

$$x' = \frac{x - VZ}{(1 - V^2/a^2)^{1/2}}, \qquad Z' = \frac{Z - Vx/a^2}{(1 - V^2/a^2)^{1/2}} \tag{4}$$

where $V = dx/dZ$, along with $y' = y$ and $z' = z$, and $0 \leq V \leq a$.

The transformation (4) is the analog to the Lorentz transformation (2) and expresses both the cosmological principle and Hubble's law for the expansion of the Universe. The relativistic factor V/a is very small and for most applications can be neglected. Also the factor Vx/a^2, appearing in the transformation for the red shift parameter in equations (4), is very small and can be neglected for most practical purposes. Accordingly in the nonrelativistic limit equations (4) yield the transformation

$$x' = x - VZ, \qquad Z' = Z \tag{5}$$

along with $y' = y$ and $z' = z$. The approximate transformation (5) is the analog to the Galilean transformation in classical mechanics, (Carmeli, 1977).

We can now proceed and express the transformation (4) in terms of the traditional variables, namely spatial coordinates and time. However, time in cosmology is not the ordinary time of special relativity. Rather, it is the cosmic time $\tau = TZ$. Equation (3) now has the form

$$x^2 + y^2 + z^2 - c^2\tau^2 = x'^2 + y'^2 + z'^2 - c^2\tau'^2 \tag{6}$$

whereas the transformation (4) becomes

$$x' = \frac{x - \tilde{v}\tau}{(1 - \tilde{v}^2/c^2)^{1/2}}, \qquad \tau' = \frac{\tau - \tilde{v}x/c^2}{(1 - \tilde{v}^2/c^2)^{1/2}} \tag{7}$$

where $\tilde{v} = T^{-1}V = T^{-1}(dx/dZ) = dx/d\tau$ is the velocity taken with respect to the cosmic time τ, along with $y' = y$ and $z' = z$. The transformation (7) is now identical in its form to the ordinary Lorentz transformation (2) except for the meaning of the time coordinate and the velocity appearing in them, with $0 \leq \tilde{v} \leq c$.

Still another form for the transformation (7) can be obtained if one assumes that the red shift of receding galaxies is proportional not only to their distances from us but also to their velocities, thus $v \propto r$, and instead of equation (6) one has

$$(v_x^2 + v_y^2 + v_z^2) - \gamma^2(x^2 + y^2 + z^2) = (v_x' + v_y'^2 + v_z'^2) - \gamma^2(x'^2 + y'^2 + z'^2) \quad (8)$$

$\gamma = T^{-1}$ is the receding parameter. If the motion is kept along the x axis as before, equation (8) then yields the transformation

$$v_x' = \frac{v_x - \alpha x}{(1 - \alpha^2/\gamma^2)^{1/2}}, \qquad x' = \frac{x - \alpha v_x/\gamma^2}{(1 - \alpha^2/\gamma^2)^{1/2}} \quad (9)$$

where α is given by $\alpha = dv_x/dx$. Thus we see that in this case the role of the spatial coordinate is taken over by the velocity and that of time by the spatial coordinate. The space now is still a Monkowskian manifold but it has six dimensions (three velocities and three spatial coordinates), and $0 \leqslant \alpha \leqslant \gamma$.

The above analysis can be generalized to spaces with constant curvatures. The simplest line element is then given by

$$ds^2 = \gamma^{-2}[(\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2] - dt^2 \quad (10)$$

where

$$\begin{aligned} \omega^1 &= \sin\theta \sin\psi \, d\phi + \cos\psi \, d\theta \\ \omega^2 &= \sin\theta \cos\psi \, d\phi - \sin\psi \, d\theta \\ \omega^3 &= \cos\theta \, d\phi + d\psi \end{aligned} \quad (11)$$

and θ, ϕ, and ψ are the Euler angles. The line element (10) can be reduced to the Einstein universe with an appropriate choice of the parameter γ. Assuming now that $d\theta = d\phi = 0$ then the analog to the Lorentz transformation is given by

$$d\psi' = \frac{d\psi - \Omega \, dt}{(1 - \Omega^2/\gamma^2)^{1/2}}, \qquad dt' = \frac{dt - \Omega \, d\psi/\gamma^2}{(1 - \Omega^2/\gamma^2)^{1/2}} \quad (12)$$

where $\Omega = d\psi/dt$, with $0 \leq \Omega \leq \gamma$.

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